DECAY RATE AND RADIAL SYMMETRY OF THE EXPONENTIAL ELLIPTIC EQUATION

KIN MING HUI AND SUNGHOON KIM

Abstract. Let $n \geq 3$, α, $\beta \in \mathbb{R}$, and let v be a solution $\Delta v + \alpha e^v + \beta x \cdot \nabla e^v = 0$ in $\mathbb{R}^n$, which satisfies the conditions $\lim_{R\to\infty} \frac{1}{\log R} \int_1^R \rho^{1-n} \left(\int_{B_\rho} e^v \, dx \right) d\rho \in (0, \infty)$ and $|x|^2 e^{v(x)} \leq A_1$ in $\mathbb{R}^n$. We prove that $\frac{v(x)}{\log |x|} \to -2$ as $|x| \to \infty$ and $\alpha > 2\beta$. As a consequence under a mild condition on v we prove that the solution is radially symmetric about the origin.

1. Introduction

In this paper we will study various properties of the solution v of the nonlinear elliptic equation

$$\Delta v + \alpha e^v + \beta x \cdot \nabla e^v = 0 \quad \text{in } \mathbb{R}^n \tag{1.1}$$

for any $n \geq 3$ where α, $\beta \in \mathbb{R}$, are some constants. Let $v = \log u$. Then u satisfies

$$\Delta \log u + \alpha u + \beta x \cdot \nabla u = 0, \quad u > 0, \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

As observed by S.Y. Hsu [Hs3], the radial symmetric solution of (1.2) is the singular limit of the radial symmetric solutions of the nonlinear elliptic equation,

$$\Delta(u^m/m) + \alpha u + \beta x \cdot \nabla u = 0, \quad u > 0, \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

as $m \searrow 0$. On the other hand as observed by P. Daskalopoulos and N. Sesum [DS], K.M. Hui and S.H. Kim [HK1], [HK2], (1.2) also arises in the study of the extinction behaviour and global behaviour of the solutions of the logarithmic diffusion equation,

$$u_t = \Delta \log u, \quad u > 0, \quad \text{in } \mathbb{R}^n. \tag{1.4}$$

(1.2) also arises in the study of self-similar solutions of (1.4) ([DS], [HK1], [HK2], [V1], [V2]). Hence in order to understand the behaviour of the solutions of (1.3) and (1.4) it is important to understand the properties of solutions of (1.1).

In [Hs2] S.Y. Hsu proved that there exists a radially symmetric solution of (1.1) (or equivalently (1.2)) if and only if either $\alpha \geq 0$ or $\beta > 0$. She also proved that when $n \geq 3$ and $\alpha > \max(2\beta, 0)$, then any radially symmetric solution v of (1.1) satisfies

$$\lim_{|x|\to\infty} |x|^2 e^{v(x)} = \frac{2(n-2)}{\alpha - 2\beta}. \tag{1.5}$$

By (1.5) and a direct computation the radially symmetric solution v of (1.1) satisfies

$$\lim_{|x|\to\infty} \frac{v(x)}{log|x|} = -2, \tag{1.6}$$

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$$A_0 := \lim_{R\to\infty}\frac{1}{\log R}\int_1^R \frac{1}{\rho^{n-1}}\left(\int_{|x|<\rho} e^v\,dx\right)d\rho\in(0,\infty) \tag{1.7}$$

and

$$|x|^2e^{v(x)}\le A_1\quad \forall x\in\mathbb{R}^n \tag{1.8}$$

for some constant $A_1>0$. A natural question is if v is a solution of (1.1) which satisfies (1.7) and (1.8) for some constant $A_1>0$, will v satisfy (1.6) and is v radially symmetric about the origin? We answer the first question in the affirmative in this paper. For the second question we prove that under some conditions on the solution v of (1.1), v is radially symmetric about the origin.

For any solution v of (1.1) we define the rotation operator Φ_{ij} by

$$\Phi_{ij}(x)=x_iv_{x_j}(x)-x_jv_{x_i}(x),\qquad \forall x=(x_1,\dots,x_n)\in\mathbb{R}^n, i\ne j, i,j=1,\cdots,n.$$

Note that if we write $x_1=\rho\cos\theta$ and $x_2=\rho\sin\theta$ where $\rho=\sqrt{x_1^2+x_2^2}$, then $\Phi_{12}(x)=\frac{\partial v}{\partial\theta}(x)$. We are now ready to state the main results of this paper.

Theorem 1.1. *Let $n\ge 3$ and $\alpha,\beta\in\mathbb{R}$. Suppose v is a solution of* (1.1) *which satisfies* (1.7) *and* (1.8) *for some constant $A_1>0$. Then v satisfies* (1.6) *and $\alpha>2\beta$.*

Corollary 1.2. *Let $n\ge 3$. Suppose $\alpha\le 2\beta$. Then* (1.1) *does not have any solution that satisfies both* (1.7) *and* (1.8) *for some constant $A_1>0$.*

Theorem 1.3. *Let $n\ge 3$ and $2\beta<\alpha<n\beta$. Suppose v is a solution of* (1.1) *which satisfies* (1.7), (1.8),

$$\|x\cdot\nabla v\|_{L^\infty(\mathbb{R}^n)}\le C<\infty. \tag{1.9}$$

and

$$\lim_{|x|\to\infty}|x|^{n-2}|\Phi_{ij}(x)|=0,\quad \forall i\ne j, i,j=1,\cdots,n. \tag{1.10}$$

Then there exists a constant $R_0>0$ such that if v is radially symmetric in B_{R_0}, then v is radially symmetric in $\mathbb{R}^n$.

Note that although there are many research done on the radial symmetry of elliptic equations without first order term by B. Gidas, W.M. Ni, and L. Nirenberg [GNN], L. Caffaralli, B. Gidas and J. Spruck [CGS], W. Chen and C. Li [CL], S.D. Taliaferro [T] and others, very little is known about the radial symmetry of elliptic equations with non-zero first order term. The reason is that one cannot use the moving plane technique to prove the radial symmetry of the solution for elliptic equations with non-zero first order term. The recent paper [KM] by E. Kamalinejad and A. Moradifam is one of the few papers that studies the radial symmetry of elliptic equations with non-zero first order term. Hence our result on radial symmetry is new.

The plan of the paper is as follows. In section two we will prove Theorem 1.1 and Corollary 1.2. In section three we will prove Theorem 1.3. For any $r>0$, $x_0\in\mathbb{R}^n$, let $B_r(x_0)=\{x\in\mathbb{R}^n : |x-x_0|<r\}$ and $B_r=B_r(0)$. Let $S^{n-1}=\{x\in\mathbb{R}^n : |x|=1\}$. We will let $n\ge 3$, $\alpha,\beta\in\mathbb{R}$, and let v be a solution of (1.1) which satisfies both (1.7) and (1.8) for some constant $A_1>0$ for the rest of the paper. We will also let A_0 be given by (1.7) for the rest of the paper.

2. Decay rate of the solution of (1.1)

In this section we will use a modification of the technique of [Hs1] to prove the decay rate (1.6) for v. We first start with a lemma.

Lemma 2.1. *For any $x_0 \in \mathbb{R}^n$, we have*

$$\lim_{R\to\infty} \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left(\int_{B_\rho(x_0)} e^v \, dx \right) d\rho = A_0. \tag{2.1}$$

Proof. Let $x_0 \in \mathbb{R}^n$ and $\rho > |x_0|$. Since $B_{\rho-|x_0|} \subset B_\rho(x_0)$, for any $R > R_0 > |x_0| + 1$,

$$\begin{aligned}
&\frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left(\int_{B_\rho(x_0)} e^v \, dx \right) d\rho \\
&\quad \geq \frac{1}{\log R} \int_{R_0}^R \frac{1}{\rho^{n-1}} \left(\int_{B_{\rho-|x_0|}} e^v \, dx \right) d\rho \\
&\quad \geq \frac{1}{\log R} \left(\frac{R_0 - |x_0|}{R_0} \right)^{n-1} \int_{R_0-|x_0|}^{R-|x_0|} \frac{1}{\rho^{n-1}} \left(\int_{B_\rho} e^v \, dx \right) d\rho \\
&\quad = -\frac{1}{\log R} \left(\frac{R_0 - |x_0|}{R_0} \right)^{n-1} \int_1^{R_0-|x_0|} \frac{1}{\rho^{n-1}} \left(\int_{B_\rho} e^v \, dx \right) d\rho \\
&\qquad + \frac{\log(R-|x_0|)}{\log R} \left(\frac{R_0 - |x_0|}{R_0} \right)^{n-1} \left[\frac{1}{\log(R-|x_0|)} \int_1^{R-|x_0|} \frac{1}{\rho^{n-1}} \left(\int_{B_\rho} e^v \, dx \right) d\rho \right].
\end{aligned}$$

Letting $R \to \infty$,

$$\liminf_{R\to\infty} \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left(\int_{B_\rho(x_0)} e^v \, dx \right) d\rho \geq \left(\frac{R_0 - |x_0|}{R_0} \right)^{n-1} A_0$$

$$\Rightarrow \liminf_{R\to\infty} \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left(\int_{B_\rho(x_0)} e^v \, dx \right) d\rho \geq A_0 \qquad \text{as } R_0 \to \infty \tag{2.2}$$

Similarly,

$$\limsup_{R\to\infty} \frac{1}{\log R} \int_1^R \frac{1}{\rho^{n-1}} \left(\int_{B_\rho(x_0)} e^v \, dx \right) d\rho \leq A_0. \tag{2.3}$$

By (2.2) and (2.3), we get (2.1) and the lemma follows. □

Lemma 2.2. *For any $x_0 \in \mathbb{R}^n$, we have*

$$\lim_{\rho\to\infty} \rho^2 \int_{|\sigma|=1} e^{v(x_0+\rho\sigma)} \, d\sigma = (n-2)A_0 \tag{2.4}$$

and

$$\lim_{\rho\to\infty} \left[\frac{1}{\log \rho} \int_{|\sigma|=1} v(x_0 + \rho\sigma) \, d\sigma \right] = -(\alpha - 2\beta)A_0. \tag{2.5}$$

Proof. Let $x_0 \in \mathbb{R}^n$ and $\rho > 0$. We first observe that (1.1) can be rewritten as

$$\Delta v + (\alpha - n\beta)e^v + \beta \, \mathrm{div}(xe^v) = 0 \quad \text{in } \mathbb{R}^n. \tag{2.6}$$

Integrating (2.6) over $B_\rho(x_0)$,

$$\begin{aligned}0 =&\rho^{n-1}\int_{|\sigma|=1}\frac{\partial v}{\partial\rho}(x_0+\rho\sigma)\,d\sigma+(\alpha-n\beta)\int_{B_\rho(x_0)}e^v\,dx+\beta\rho^n\int_{|\sigma|=1}e^{v(x_0+\rho\sigma)}\,d\sigma\\&+\beta\rho^{n-1}\int_{|\sigma|=1}e^{v(x_0+\rho\sigma)}\,(x_0\cdot\sigma)\,d\sigma.\end{aligned}\tag{2.7}$$

Let $R>R_0\ge 1$. Dividing (2.7) by ρ^{n-1} and integrating over $\rho\in(1,R)$,

$$\begin{aligned}&\int_{|\sigma|=1}v(x_0+R\sigma)\,d\sigma-\int_{|\sigma|=1}v(x_0+\sigma)\,d\sigma\\&\qquad=-(\alpha-n\beta)\int_1^R\frac{1}{\rho^{n-1}}\left(\int_{B_\rho(x_0)}e^v\,dx\right)d\rho-\beta\int_1^R\rho\left(\int_{|\sigma|=1}e^{v(x_0+\rho\sigma)}\,d\sigma\right)d\rho\\&\qquad\quad-\beta\int_1^R\left(\int_{|\sigma|=1}e^{v(x_0+\rho\sigma)}(x_0\cdot\sigma)\,d\sigma\right)d\rho.\end{aligned}\tag{2.8}$$

Let $\{\rho_i\}_{i=1}^\infty$ be a sequence such that $\rho_i>2|x_0|+1$ for all $i\in\mathbb{Z}^+$ and $\rho_i\to\infty$ as $i\to\infty$. Then by (1.8) there exists a constant $C_1>0$ such that

$$\rho_i^2\int_{|\sigma|=1}e^{v(x_0+\rho_i\sigma)}\,d\sigma\le C_1\quad\forall i\in\mathbb{Z}^+.$$

Hence the sequence $\{\rho_i\}_{i=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself such that

$$\rho_i^2\int_{|\sigma|=1}e^{v(x_0+\rho_i\sigma)}\,d\sigma$$

converges to some non-negative number as $i\to\infty$. On the other hand by (2.1), there exists a constant $R_1>1$ such that

$$\begin{aligned}&\frac{A_0}{2}\le\frac{1}{\log R}\int_1^R\frac{1}{\rho^{n-1}}\left(\int_{B_\rho(x_0)}e^v\,dx\right)d\rho\le\frac{1}{(n-2)\log R}\int_{B_R(x_0)}e^v\,dx\quad\forall R\ge R_1\\&\Rightarrow\quad\int_{B_\rho(x_0)}e^v\,dx\ge\frac{(n-2)}{2}A_0\log\rho\to\infty\qquad\text{as }\rho\to\infty.\end{aligned}\tag{2.9}$$

Then by (1.7), (2.1), (2.9), and the l'Hospital rule,

$$\begin{aligned}A_0&=\lim_{\rho\to\infty}\frac{\frac{1}{\rho_i^{n-1}}\int_{B_{\rho_i}(x_0)}e^v\,dx}{\frac{1}{\rho_i}}=\lim_{i\to\infty}\frac{1}{\rho_i^{n-2}}\int_{B_{\rho_i}(x_0)}e^v\,dx\\&=\lim_{i\to\infty}\frac{\rho_i^{n-1}\int_{|\sigma|=1}e^{v(x_0+\rho_i\sigma)}\,d\sigma}{(n-2)\rho_i^{n-3}}=\frac{1}{(n-2)}\lim_{i\to\infty}\rho_i^2\int_{|\sigma|=1}e^{v(x_0+\rho_i\sigma)}\,d\sigma\end{aligned}$$

Since the sequence $\{\rho_i\}_{i=1}^\infty$ is arbitrary, (2.4) follows. Then by (2.4) for any $0<\varepsilon<1$ there exists a constant $R_0>1$ such that

$$\left|\rho^2\int_{|\sigma|=1}e^{v(x_0+\rho\sigma)}d\sigma-(n-2)A_0\right|<(n-2)A_0\varepsilon\qquad\forall\rho\ge R_0.\tag{2.10}$$

By (2.10),

$$\begin{aligned}&\int_1^{R_0}\rho\left(\int_{|\sigma|=1}e^{v(x_0+\rho\sigma)}\,d\sigma\right)d\rho+(1-\varepsilon)(n-2)A_0\log(R/R_0)\\ &\qquad\le\int_1^{R}\rho\left(\int_{|\sigma|=1}e^{v(x_0+\rho\sigma)}\,d\sigma\right)d\rho\\ &\qquad\quad\le\int_1^{R_0}\rho\left(\int_{|\sigma|=1}e^{v(x_0+\rho\sigma)}\,d\sigma\right)d\rho+(1+\varepsilon)(n-2)A_0\log(R/R_0)\quad\forall R>R_0\end{aligned}\tag{2.11}$$

and

$$\begin{aligned}&\left|\int_1^{R}\int_{|\sigma|=1}e^{v(x_0+\rho\sigma)}(x_0\cdot\sigma)\,d\sigma d\rho\right|\\ &\qquad\le|x_0|\int_1^{R_0}\left(\int_{|\sigma|=1}e^{v(x_0+\rho\sigma)}\,d\sigma\right)d\rho+2(n-2)A_0|x_0|\left(\frac{1}{R_0}-\frac{1}{R}\right)\end{aligned}\tag{2.12}$$

for any $R>R_0$. Dividing (2.11) and (2.12) by $\log R$ and letting first $R\to\infty$ and then $\varepsilon\to0$, we get

$$\lim_{R\to\infty}\frac{1}{\log R}\int_1^{R}\rho\left(\int_{|\sigma|=1}e^{v(x_0+\rho\sigma)}\,d\sigma\right)d\rho=(n-2)A_0\tag{2.13}$$

and

$$\lim_{R\to\infty}\frac{1}{\log R}\int_1^{R}\int_{|\sigma|=1}e^{v(x_0+\rho\sigma)}(x_0\cdot\sigma\,d\sigma d\rho=0.\tag{2.14}$$

Dividing (2.8) by $\log R$ and letting $R\to\infty$, by (1.7), (2.13) and (2.14), we get (2.5) and the lemma follows. □

We now let

$$\begin{cases}w_1(x)=\dfrac{1}{n(2-n)\omega_n}\displaystyle\int_{\mathbb{R}^n}\left(\frac{1}{|x-y|^{n-2}}-\frac{1}{|y|^{n-2}}\right)e^{v(y)}\,dy & \forall x\in\mathbb{R}^n\\ w_{2,R}(x)=\dfrac{1}{n(2-n)\omega_n}\displaystyle\int_{|y|\le R}\left(\frac{1}{|x-y|^{n-2}}-\frac{1}{|y|^{n-2}}\right)\operatorname{div}\left(e^{v(y)}y\right)\,dy & \forall x\in B_R,\ R>0\end{cases}\tag{2.15}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Then w_1 and $w_{2,R}$ are well-defined in $\mathbb{R}^n$ and B_R respectively with $w_1\in C(\mathbb{R}^n)$ and $w_{2,R}\in C(B_R)$. Moreover

$$\begin{cases}\triangle w_1=e^v & \text{in }\mathbb{R}^n\\ \triangle w_{2,R}=\operatorname{div}\left(e^{v(y)}y\right) & \text{in }B_R\quad\forall R>0.\end{cases}\tag{2.16}$$

Lemma 2.3. *As $R\to\infty$, $w_{2,R}$ will converge uniformly on every compact subset of $\mathbb{R}^n$ to*

$$\begin{aligned}w_2(x)&=\frac{1}{n(n-2)\omega_n}\int_{\mathbb{R}^n}e^{v(y)}\nabla_y\left(\frac{1}{|x-y|^{n-2}}-\frac{1}{|y|^{n-2}}\right)\cdot y\,dy\\ &=\frac{1}{n\omega_n}\int_{\mathbb{R}^n}\left(\frac{(x-y)\cdot y}{|x-y|^n}+\frac{1}{|y|^{n-2}}\right)e^{v(y)}\,dy.\end{aligned}\tag{2.17}$$

Proof. Let $R_0 > 1$, $R > 2R_0$ and $|x| \le R_0$. Now

$$\begin{aligned} w_{2,R}(x) =& \frac{1}{n(2-n)\omega_n}\int_{|y|=R}\left(\frac{1}{|x-y|^{n-2}}-\frac{1}{|y|^{n-2}}\right)e^{v(y)}y\cdot\nu(y)\,d\sigma_R(y)\\ &+\frac{1}{n(n-2)\omega_n}\int_{|y|<R}e^{v(y)}\nabla_y\left(\frac{1}{|x-y|^{n-2}}-\frac{1}{|y|^{n-2}}\right)\cdot y\,dy\\ =& I_{1,R}(x)+I_{2,R}(x)\end{aligned} \tag{2.18}$$

where $\nu(y) = \frac{y}{|y|}$. Since there exists a constant $C_1 > 0$ such that

$$\begin{aligned}\left|\frac{1}{|x-y|^{n-2}}-\frac{1}{|y|^{n-2}}\right| &= \left|\int_0^1\frac{\partial}{\partial\theta}\left(\frac{1}{|\theta x-y|^{n-2}}\right)d\theta\right| = (n-2)\left|\int_0^1\frac{(\theta x-y)\cdot x}{|\theta x-y|^n}\,d\theta\right|\\ &\le C_1\frac{|x|}{|y|^{n-1}}\qquad\forall |y|\ge 2|x|,\end{aligned} \tag{2.19}$$

by (2.10) there exists a constant $C_2 > 0$ such that

$$\sup_{|x|<R_0}|I_{1,R}(x)| \le \frac{C_2R_0}{R}\to 0\qquad\text{as } R\to\infty. \tag{2.20}$$

By the Taylor expansion (P.231 of [GNN]),

$$\frac{1}{|x-y|^n} = \frac{1}{|y|^n}\left(1+\frac{n}{|y|^2}\sum_{j=1}^n x_jy_j + O\left(\frac{1}{|y|^2}\right)\right)\quad \forall |y|\ge 2|x|,\ x=(x_1\cdots,x_n),\ y=(y_1,\cdots,y_n).$$

Hence there exists a constant $C_3 > 0$ such that

$$\begin{aligned}&\left|\nabla_y\left(\frac{1}{|x-y|^{n-2}}-\frac{1}{|y|^{n-2}}\right)\cdot y\right|\\ &\quad= (n-2)\left|\frac{(x-y)\cdot y}{|x-y|^n}+\frac{1}{|y|^{n-2}}\right|\\ &\quad= (n-2)\left|\frac{x\cdot y}{|y|^n}\left(1+\frac{n}{|y|^2}\sum_{j=1}^n x_jy_j+\cdots\right)-\frac{1}{|y|^{n-2}}\left(\frac{n}{|y|^2}\sum_{j=1}^n x_jy_j+\cdots\right)\right|\\ &\quad\le C_3\frac{|x|}{|y|^{n-1}},\qquad \forall |y|\ge 2|x|,\ x=(x_1\cdots,x_n),\ y=(y_1,\cdots,y_n).\end{aligned} \tag{2.21}$$

By (2.10) and (2.21),

$$\begin{aligned}&\int_{2R_0<|y|<R}e^{v(y)}\left|\nabla_y\left(\frac{1}{|x-y|^{n-2}}-\frac{1}{|y|^{n-2}}\right)\cdot y\right|dy\\ &\quad<\int_{2R_0}^R\frac{C_3R_0}{\rho^2}\left(\rho^2\int_{|\sigma|=1}e^{v(\rho\sigma)}\,d\sigma\right)d\rho\le C'R_0\int_{2R_0}^R\frac{1}{\rho^2}\,d\rho\le C''<\infty\end{aligned} \tag{2.22}$$

holds for any $|x| \le R_0$ and $R > 2R_0$. On the other hand since v is continuous and v satisfies (1.8), $e^v \in L^\infty(\mathbb{R}^n)$. Hence by (2.21),

$$\begin{aligned}&\int_{|y|<2R_0}e^{v(y)}\left|\nabla_y\left(\frac{1}{|x-y|^{n-2}}-\frac{1}{|y|^{n-2}}\right)\cdot y\right|dy\\ &\qquad\le C_3R_0\int_{|y|<3R_0}\frac{1}{|y|^{n-1}}\,dy = 3n\omega_nC_3R_0^2<\infty.\end{aligned} \tag{2.23}$$

By (2.22) and (2.23) and the Lebesgue Dominated Convergence Theorem,

$$I_{2,R}(x) \to \frac{1}{n\omega_n}\int_{\mathbb{R}^n}\left(\frac{(x-y)\cdot y}{|x-y|^n}+\frac{1}{|y|^{n-2}}\right)e^{v(y)}\,dy \tag{2.24}$$

uniformly on $\{|x|\le R_0\}$ as $R\to\infty$. By (2.18), (2.20) and (2.24), the lemma follows. $\square$

Lemma 2.4. *There exists a constant $C>0$ such that*

$$|w_1(x)|\le C\log|x| \qquad \forall |x|\ge 2. \tag{2.25}$$

Proof. Let $|x|\ge 2$. We first split w_1 into two parts as follows.

$$\begin{aligned} w_1(x) &= \frac{1}{n(2-n)\omega_n}\int_{|y|>2|x|}\left(\frac{1}{|x-y|^{n-2}}-\frac{1}{|y|^{n-2}}\right)e^{v(y)}\,dy\\ &\quad+\frac{1}{n(2-n)\omega_n}\int_{|y|\le 2|x|}\frac{e^{v(y)}}{|x-y|^{n-2}}\,dy+\frac{1}{n(n-2)\omega_n}\int_{|y|\le 2|x|}\frac{e^{v(y)}}{|y|^{n-2}}\,dy\\ &:= I_1+I_2+I_3. \end{aligned} \tag{2.26}$$

By (2.10),

$$\left|\int_{|\sigma|=1}e^{v(\rho\sigma)}\,d\sigma\right|\le\frac{C}{\rho^2}, \qquad \forall \rho>0 \tag{2.27}$$

for some constant $C>0$. Hence by (2.19) and (2.27),

$$|I_1|\le C\int_{2|x|}^{\infty}\frac{|x|}{\rho^{n-1}}\cdot\frac{1}{\rho^2}\cdot\rho^{n-1}\,d\rho=C'<\infty \quad \forall x\in\mathbb{R}^n. \tag{2.28}$$

for some constant $C'>0$. By (2.27),

$$\begin{aligned} I_3 &= \frac{1}{n(n-2)\omega_n}\int_{0<|y|\le 1}\frac{e^{v(y)}}{|y|^{n-2}}\,dy+\frac{1}{n(n-2)\omega_n}\int_{1\le|y|\le 2|x|}\frac{e^{v(y)}}{|y|^{n-2}}\,dy\\ &\le C\int_0^1\frac{1}{\rho^{n-2}}\cdot\rho^{n-1}\,d\rho+C\int_1^{2|x|}\frac{1}{\rho^{n-2}}\cdot\frac{1}{\rho^2}\cdot\rho^{n-1}\,d\rho\\ &= C(1+\log(2|x|))\\ &\le C'\log|x| \qquad\qquad \forall |x|\ge 2. \end{aligned} \tag{2.29}$$

On the other hand by (1.8),

$$|I_2|\le\int_{D_1(x)}\frac{A_1}{|x-y|^{n-2}|y|^2}\,dy+\int_{D_2(x)}\frac{A_1}{|x-y|^{n-2}|y|^2}\,dy=I_{2,1}+I_{2,2} \tag{2.30}$$

where

$$\begin{cases} D_1(x)=\{y\in\mathbb{R}^n:|y|\le 2|x| \quad\text{and}\quad |x-y|\le\frac{|x|}{2}\}\\ D_2(x)=\{y\in\mathbb{R}^n:|y|\le 2|x| \quad\text{and}\quad |x-y|\ge\frac{|x|}{2}\}. \end{cases} \tag{2.31}$$

Since

$$|x-y|\le\frac{|x|}{2} \quad\Rightarrow\quad |y|\ge\frac{|x|}{2},$$

we have

$$I_{2,1}\le\frac{C}{|x|^2}\int_{|x-y|\le\frac{|x|}{2}}\frac{1}{|x-y|^{n-2}}\,dy=C'<\infty \quad \forall x\ne 0 \tag{2.32}$$

for some constant $C'>0$. Finally on $D_2(x)$,

$$I_{2,2}\le\frac{C}{|x|^{n-2}}\int_{|y|\le 2|x|}\frac{1}{|y|^2}\,dy=C''<\infty\quad\forall x\ne 0 \tag{2.33}$$

for some constant $C''>0$. By (2.26), (2.28), (2.29), (2.30), (2.32) and (2.33), we get (2.25) and lemma follows. □

Lemma 2.5. *The following holds.*

$$v(x)+(\alpha-n\beta)w_1(x)+\beta w_2(x)=v(0)\quad\forall\mathbb{R}^n.$$

Proof. Let $q=v+(\alpha-n\beta)w_1+\beta w_2$. Then by (2.16) and Lemma 2.3,

$$\triangle q=0\quad\text{ in }\mathbb{R}^n\quad\text{ in the distribution sense.}$$

Let $x_0,x_1\in\mathbb{R}^n$. By (1.8) there exists a constant $R_0>0$ such that

$$v(x)<0\qquad\forall|x|\ge R_0. \tag{2.34}$$

By the mean value theorem for harmonic functions,

$$q(x_0)-q(x_1)=\frac{1}{|B_R|}\left(\int_{B_R(x_0)}q\,dx-\int_{B_R(x_1)}q\,dx\right)\qquad\forall R>0. \tag{2.35}$$

Let $a=|x_0-x_1|$ and $R>R_0+2+2a+|x_0|+|x_1|$. Since $B_{R_0}\subset B_{R-2a}(x_1)\subset B_{R-a}(x_0)\subset B_R(x_1)$,

$$\mathbb{R}^n\setminus B_{R-a}(x_0)\subset\mathbb{R}^n\setminus B_{R-2a}(x_1)\subset\mathbb{R}^n\setminus B_{R_0}. \tag{2.36}$$

Then by (2.34), (2.36) and Lemma 2.2,

$$\begin{aligned}
&\frac{1}{|B_R|}\left(\int_{B_R(x_0)}v\,dx-\int_{B_R(x_1)}v\,dx\right)\\
&\quad=\frac{1}{|B_R|}\left(\int_{B_R(x_0)\setminus B_{R-a}(x_0)}v\,dx-\int_{B_R(x_1)\setminus B_{R-a}(x_0)}v\,dx\right)\\
&\quad\le-\frac{1}{|B_R|}\int_{B_R(x_1)\setminus B_{R-a}(x_0)}v\,dx\\
&\quad\le-\frac{1}{|B_R|}\int_{B_R(x_1)\setminus B_{R-2a}(x_1)}v\,dx\\
&\quad\le-\frac{1}{|B_R|}\left(\inf_{R-2a\le\rho\le R}\left[\frac{1}{\log\rho}\int_{|\sigma|=1}v(x_1+\rho\sigma)\,d\sigma\right]\right)\int_{R-2a}^{R}\rho^{n-1}\log\rho\,d\rho\\
&\quad\le-\frac{2aR^{n-1}\log R}{|B_R|}\left(\inf_{R-2a\le\rho\le R}\left[\frac{1}{\log\rho}\int_{|\sigma|=1}v(x_1+\rho\sigma)\,d\sigma\right]\right)\\
&\quad\to 0\qquad\qquad\text{as }R\to\infty.
\end{aligned}\tag{2.37}$$

On the other hand, since $(B_R(x_0)\setminus B_R(x_1))\cup(B_R(x_1)\setminus B_R(x_0))\subset(B_{R+a}(x_0)\setminus B_{R-a}(x_0))$,

$$\begin{aligned}
&\frac{1}{|B_R|}\left(\int_{B_R(x_0)}((\alpha-n\beta)w_1+\beta w_2)\,dx-\int_{B_R(x_1)}((\alpha-n\beta)w_1+\beta w_2)\,dx\right)\\
&\qquad\le\frac{|\alpha-n\beta|}{|B_R|}\int_{B_{R+a}(x_0)\setminus B_{R-a}(x_0)}|w_1|\,dx+\frac{|\beta|}{|B_R|}\int_{B_{R+a}(x_0)\setminus B_{R-a}(x_0)}|w_2|\,dx\\
&\qquad:=|\alpha-n\beta|J_1+|\beta|J_2.
\end{aligned}\tag{2.38}$$

By Lemma 2.4,

$$J_1 \le \frac{C_1}{|B_R|}\int_{R-a\le|x-x_0|\le R+a}\log|x|\,dx \le \frac{2C_1 a\log(|x_0|+R+a)|\partial B_{R+a}|}{|B_R|}\to 0 \qquad \text{as } R\to\infty \tag{2.39}$$

for some constant $C_1>0$. We next observe that by (2.17),

$$\begin{aligned} J_2 \le & \frac{1}{n\omega_n|B_R|}\int_{B_{R+a}(x_0)\setminus B_{R-a}(x_0)}\left|\int_{|y|>2|x|}\left(\frac{(x-y)\cdot y}{|x-y|^n}+\frac{1}{|y|^{n-2}}\right)e^{v(y)}\,dy\right|\,dx \\ & + \frac{1}{n\omega_n|B_R|}\int_{B_{R+a}(x_0)\setminus B_{R-a}(x_0)}\left[\int_{|y|\le 2|x|}\frac{|y|}{|x-y|^{n-1}}e^{v(y)}\,dy+\int_{|y|\le 2|x|}\frac{e^{v(y)}}{|y|^{n-2}}\,dy\right]\,dx \\ := & J_{2,1}+J_{2,2}. \end{aligned} \tag{2.40}$$

By (2.21) and (2.27),

$$\begin{aligned} J_{2,1} \le & \frac{C}{|B_R|}\int_{B_{R+a}(x_0)\setminus B_{R-a}(x_0)}|x|\left(\int_{2|x|}^{\infty}\frac{1}{\rho^{n-1}}\cdot\frac{1}{\rho^2}\cdot\rho^{n-1}\,d\rho\right)dx \\ = & \frac{C}{2|B_R|}\,|B_{R+a}(x_0)\setminus B_{R-a}(x_0)| \\ \le & C'\frac{(R+a)^{n-1}}{R^n}\to 0 \qquad \text{as } R\to\infty. \end{aligned} \tag{2.41}$$

Let

$$\widetilde{I}_1=\int_{|y|\le 2|x|}\frac{|y|}{|x-y|^{n-1}}e^{v(y)}\,dy \qquad \text{and} \qquad \widetilde{I}_2=\int_{|y|\le 2|x|}\frac{e^{v(y)}}{|y|^{n-2}}\,dy.$$

Then by the proof of Lemma 2.4,

$$\widetilde{I}_2\le C\log|x| \qquad \forall |x|\ge 2. \tag{2.42}$$

On the other hand by (1.8),

$$\begin{aligned} \widetilde{I}_1 \le & \int_{|y|\le 2|x|}\frac{A_1}{|x-y|^{n-1}|y|}\,dy=\int_{D_1(x)}\frac{A_1}{|x-y|^{n-1}|y|}\,dy+\int_{D_2(x)}\frac{A_1}{|x-y|^{n-1}|y|}\,dy \\ = & \widetilde{I}_{1,1}+\widetilde{I}_{1,2} \end{aligned} \tag{2.43}$$

where $D_1(x)$, $D_2(x)$, are as given by (2.31). Since $|y|\ge|x|/2$ for all $y\in D_1(x)$,

$$\widetilde{I}_{1,1}\le\frac{C}{|x|}\int_{|x-y|\le\frac{|x|}{2}}\frac{1}{|x-y|^{n-1}}\,dy=C'<\infty \tag{2.44}$$

and

$$\widetilde{I}_{1,2}\le\frac{2^{n-1}A_1}{|x|^{n-1}}\int_{|y|\le 2|x|}\frac{1}{|y|}\,dy=C''<\infty \tag{2.45}$$

for some constants $C'>0$, $C''>0$. By (2.42), (2.43), (2.44) and (2.45),

$$\begin{aligned} J_{2,2} \le & \frac{C}{|B_R|}\int_{B_{R+a}(x_0)\setminus B_{R-a}(x_0)}(1+\log|x|)\,dx \\ \le & C(1+\log(|x_0|+R+a))\frac{|B_{R+a}(x_0)\setminus B_{R-a}(x_0)|}{|B_R|} \\ \le & C'\frac{(R+a)^{n-1}\log(a+|x_0|+R)}{R^n}\to 0 \qquad \text{as } R\to\infty. \end{aligned} \tag{2.46}$$

Letting $R \to 0$ in (2.35), by (2.37), (2.38), (2.39), (2.40), (2.41) and (2.46), we get

$$q(x_0) - q(x_1) \le 0 \qquad \forall x_0,\, x_1 \in \mathbb{R}^n.$$

Since x_0, x_1 are arbitrary, by interchanging the roles of x_0 and x_1 in the above argument we get

$$q(x_1) - q(x_0) \le 0 \qquad \forall x_0,\, x_1 \in \mathbb{R}^n.$$

Hence

$$q(x_0) - q(x_1) = 0 \qquad \forall x_0,\, x_1 \in \mathbb{R}^n.$$

Thus q is a constant. Hence $q(x) = q(0) = v(0)$ for any $x \in \mathbb{R}^n$ and the lemma follows. □

Lemma 2.6.

$$\frac{w_1(x)}{\log|x|} \to \frac{A_0}{n\omega_n} \qquad \textit{as } |x| \to \infty.$$

Proof. Let I_1, I_2, and I_3 be the same as the proof of Lemma 2.4. Then I_1 satisfies (2.28). By the proof of Lemma 2.4 there exists a constant $C_1 > 0$ such that

$$|I_2| \le C_1 \quad \forall x \in \mathbb{R}^n. \tag{2.47}$$

By Lemma 2.2 there exists a constant $R_0 > 1$ such that (2.10) holds with $\varepsilon = 1/3$. Now

$$\begin{aligned} I_3 &= \frac{1}{n(n-2)\omega_n}\int_{|y|\le|x|}\frac{e^{v(y)}}{|y|^{n-2}}\,dy + \frac{1}{n(n-2)\omega_n}\int_{|x|\le|y|\le 2|x|}\frac{e^{v(y)}}{|y|^{n-2}}\,dy \\ &= I_{3,1} + I_{3,2}. \end{aligned} \tag{2.48}$$

Since by (2.10) for any $|x| > R_0$,

$$I_{3,1} \ge \frac{1}{n(n-2)\omega_n}\int_{R_0}^{|x|}\rho\int_{|\sigma|=1}e^{v(\rho\sigma)}\,d\sigma\,d\rho \ge C\int_{R_0}^{|x|}\frac{d\rho}{\rho} = C\log\left(\frac{|x|}{R_0}\right) \to \infty \quad \text{as } |x| \to \infty, \tag{2.49}$$

by (2.4) and l'Hospital rule,

$$\lim_{|x|\to\infty}\frac{I_{3,1}}{\log|x|} = \frac{\lim_{|x|\to\infty}|x|^2\int_{|\sigma|=1}e^{v(|x|\sigma)}\,d\sigma}{n(n-2)\omega_n} = \frac{A_0}{n\omega_n}. \tag{2.50}$$

By (2.10),

$$I_{3,2} = \frac{1}{n(n-2)\omega_n}\int_{|x|}^{2|x|}\rho\left(\int_{|\sigma|=1}e^{v(\rho\sigma)}\,d\sigma\right)d\rho \le C_2\int_{|x|}^{2|x|}\frac{d\rho}{\rho} = C_2\log 2 < \infty \quad \forall |x| > R_0, \tag{2.51}$$

for some constant $C_2 > 0$. By (2.48), (2.50) and (2.51),

$$\lim_{|x|\to\infty}\frac{I_3}{\log|x|} = \frac{A_0}{n\omega_n}. \tag{2.52}$$

Hence by (2.28), (2.47) and (2.52), we get

$$\lim_{|x|\to\infty}\frac{w_1(x)}{\log|x|} = \frac{A_0}{n\omega_n}$$

and lemma the follows. □

Lemma 2.7.

$$\frac{w_2(x)}{\log|x|} \to \frac{(n-2)A_0}{n\omega_n} \qquad \textit{as } |x| \to \infty. \tag{2.53}$$

Proof. By (2.17),

$$
\begin{aligned}
w_2(x) &= \frac{1}{n\omega_n}\int_{|y|\le 2|x|}\frac{(x-y)\cdot y}{|x-y|^n}\cdot e^{v(y)}\,dy + \frac{1}{n\omega_n}\int_{|y|>2|x|}\left(\frac{(x-y)\cdot y}{|x-y|^n}+\frac{1}{|y|^{n-2}}\right)e^{v(y)}\,dy\\
&\qquad + \frac{1}{n\omega_n}\int_{|y|\le 2|x|}\frac{e^{v(y)}}{|y|^{n-2}}\,dy\\
&= I_1 + I_2 + (n-2)I_3.
\end{aligned}
\tag{2.54}
$$

Since

$$|I_1| \le \frac{1}{n\omega_n}\int_{|y|\le 2|x|}\frac{|y|}{|x-y|^{n-1}}e^{v(y)}\,dy,$$

by the proof of Lemma 2.5 there exists a constant $C>0$ such that

$$|I_1| \le C \quad \forall x\in\mathbb{R}^n. \tag{2.55}$$

By (2.21) and (2.27),

$$|I_2| \le C|x|\int_{|y|\ge 2|x|}\frac{e^{v(y)}}{|y|^{n-1}}\,dy \le C'|x|\int_{2|x|}^{\infty}\frac{d\rho}{\rho^2} = C'' < \infty \tag{2.56}$$

for some constants $C'>0$, $C''>0$. By the proof of Lemma 2.6, I_3 satisfies (2.52). By (2.52), (2.54), (2.55) and (2.56), we get (2.53) and the lemma follows. □

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.5, Lemma 2.6 and Lemma 2.7,

$$\lim_{|x|\to\infty}\frac{v(x)}{\log|x|} = -L$$

where

$$L = \frac{(\alpha-2\beta)A_0}{n\omega_n}.$$

Suppose that $L>2$. Then there exist constants $a\in(2,n)$ and $R_1>1$ such that

$$e^{v(x)} \le |x|^{-a} \qquad \forall |x|\ge R_1.$$

Hence

$$
\begin{aligned}
&0 < \frac{1}{\log R}\int_1^R \frac{1}{\rho^{n-1}}\left(\int_{B_\rho} e^v\,dy\right)d\rho \le \frac{C}{\log R}, \qquad \forall R>R_1\\
&\Rightarrow \lim_{R\to\infty}\frac{1}{\log R}\int_1^R\frac{1}{\rho^{n-1}}\left(\int_{B_\rho}e^v\,dy\right)d\rho = 0 \quad \text{ as } R\to\infty
\end{aligned}
$$

which contradicts (1.7). Hence $L\le 2$. Suppose that $L<2$. Then there exist constants $b\in(0,2)$ and $R_2>1$ such that

$$e^{v(x)} \ge |x|^{-b} \qquad \forall |x|\ge R_2. \tag{2.57}$$

By (2.57) and a direct computation,

$$\frac{1}{\log R}\int_1^R \frac{1}{\rho^{n-1}}\left(\int_{B_\rho} e^v\,dy\right)d\rho \ge \frac{C_1R^{2-b}}{\log R}-\frac{C_2}{\log R}\qquad \forall R>R_2$$
$$\Rightarrow\quad \lim_{R\to\infty}\frac{1}{\log R}\int_1^R \frac{1}{\rho^{n-1}}\left(\int_{B_\rho} e^v\,dy\right)d\rho=\infty\quad \text{ as } R\to\infty$$

which again contradicts (1.7). Hence $L=2$ and the theorem follows. □

Corollary 2.8. *If v is a solution of* (1.1) *which satisfies* (1.7) *and* (1.8), *then*

$$\alpha>2\beta \qquad \text{and} \qquad A_0=\frac{2n\omega_n}{\alpha-2\beta}.$$

3. Radial symmetry of the solution

In this section we will prove that under some condition on the solution v of (1.1), v is radially symmetric about the origin. We first start with a proposition.

Proposition 3.1. *Let $n\ge 3$, $\alpha>2\beta$, $a_0\in\mathbb{R}$, and let v be the unique radially symmetric solution of* (1.1) *with $v(0)=a_0$ constructed in* [Hs2]. *Then* (1.5) *holds and*

$$\lim_{r\to\infty} rv'(r)=-2.$$

Proof. We first observe that (1.5) is proved in [Hs2]. We next observe that by putting $x_0=0$ in (2.7) and simplifying,

$$rv'(r)=\frac{n\beta-\alpha}{r^{n-2}}\int_0^r \rho^{n-1}e^{v(\rho)}\,d\rho-\beta r^2e^{v(r)}.$$

Hence by (1.5),

$$\begin{aligned}\lim_{r\to\infty} rv'(r)=&\lim_{r\to\infty}\frac{n\beta-\alpha}{r^{n-2}}\int_0^r \rho^{n-1}e^{v(\rho)}\,d\rho-\beta\lim_{r\to\infty}r^2e^{v(r)}\\ =&(n\beta-\alpha)\lim_{r\to\infty}\frac{r^{n-1}e^{v(r)}}{(n-2)r^{n-3}}-\beta\lim_{r\to\infty}r^2e^{v(r)}\\ =&\left[\frac{n\beta-\alpha}{n-2}-\beta\right]\lim_{r\to\infty}r^2e^{v(r)}\\ =&-2.\end{aligned}$$

□

Proposition 3.2. *Let $n\ge 3$, $\alpha>2\beta$, and let v be a solution of* (1.1) *which satisfies* (1.7), (1.8) *and* (1.9). *Then*

$$x\cdot\nabla v\to -2 \qquad \text{uniformly as } |x|\to\infty.$$

Proof. Let

$$V(x)=x\cdot\nabla v(x).$$

By (1.1) and a direct computation V satisfies

$$\triangle V=F(x)\qquad \text{in } \mathbb{R}^n \tag{3.1}$$

where

$$F(x)\equiv -\left(2\alpha+(\alpha+2\beta)V+\beta V^2\right)e^v-\beta(x\cdot\nabla V)e^v. \tag{3.2}$$

Let $0 < \mu < 1$ and $x \in \mathbb{R}^n$ be such that $|x| > 1$. For any $y \in B_{\frac{|x|}{2}}(x)$, let

$$d_y = \operatorname{dist}\left(y, \partial B_{\frac{|x|}{2}}(x)\right) \qquad \text{and} \qquad R = \frac{1}{3}d_y.$$

Then by Green's representation formula,

$$V(z) = H(z) + N(z) \qquad \forall z \in B_{2\mu R}(y) \tag{3.3}$$

for some harmonic function H in $B_{2\mu R}(y)$ where

$$N(z) = \frac{1}{n(2-n)\omega_n}\int_{B_{2\mu R}(y)} |z-w|^{2-n}F(w)\,dw \tag{3.4}$$

is the Newtonian potential of F in $B_{2\mu R}(y)$. Since $d_z \geq d_y/3 = R$ for any $z \in B_{2\mu R}(y)$, by (3.3), (3.4), and Theorem 2.1 and Lemma 4.1 of [GT],

$$\begin{aligned} d_y|\nabla N(y)| =& \frac{d_y}{n(n-2)\omega_n}\left|\int_{B_{2\mu R}(y)} \nabla_y(|y-w|^{2-n})F(w)\,dw\right| \\ \leq& C_1\mu d_y^2 \sup_{B_{2\mu R}(y)}|F| \\ \leq& C_1\mu\|d_z^2F(z)\|_{L^\infty(B_{|x|/2}(x))} \end{aligned}$$

and

$$\begin{aligned} d_y|\nabla H(y)| \leq& \frac{C_2 d_y}{\mu R}\sup_{B_{2\mu R}(y)}|H| \leq \frac{3C_2}{\mu}\left(\sup_{B_{2\mu R}(y)}|V| + \mu^2R^2\sup_{B_{2\mu R}(y)}|F|\right) \\ \leq& \frac{3C_2}{\mu}\left(\|V\|_{L^\infty(B_{|x|/2}(x))} + \mu^2\left\|d_z^2F(z)\right\|_{L^\infty(B_{|x|/2}(x))}\right) \end{aligned}$$

for some constants $C_1 > 0$, $C_2 > 0$. Hence

$$\|d_y\nabla V(y)\|_{L^\infty(B_{|x|/2}(x))} \leq \frac{C_3}{\mu}\left(\|V\|_{L^\infty(B_{|x|/2}(x))} + \mu^2\|d_z^2F(z)\|_{L^\infty(B_{|x|/2}(x))}\right) \tag{3.5}$$

for some constant $C_3 > 0$. Since $d_z \leq |x|/2 \leq |z|$ for any $z \in B_{|x|/2}(x)$,

$$d_z^2 e^{v(z)} \leq A_1 \quad \forall z \in B_{|x|/2}(x). \tag{3.6}$$

Then by (1.8), (3.2) and (3.6),

$$|d_z^2F(z)| \leq C_4(1 + \|V\|_{L^\infty(B_{|x|/2}(x))} + \|V\|^2_{L^\infty(B_{|x|/2}(x))} + \|d_w\nabla V(w)\|_{L^\infty(B_{|x|/2}(x))}) \tag{3.7}$$

for some constant $C_4 > 0$ and any $z \in B_{|x|/2}(x)$. Hence by (3.5) and (3.7),

$$\begin{aligned} &\|d_y\nabla V(y)\|_{L^\infty(B_{|x|/2}(x))} \\ &\leq \frac{C_5}{\mu}(1 + \|V\|_{L^\infty(B_{|x|/2}(x))} + \|V\|^2_{L^\infty(B_{|x|/2}(x))}) + \mu^2\|d_y\nabla V(y)\|_{L^\infty(B_{|x|/2}(x))} \end{aligned} \tag{3.8}$$

for some constant $C_5 > 0$. We now choose $\mu = 1/(2C_5)$. Then by (3.8),

$$\frac{|x|}{4}\|\nabla V\|_{L^\infty(B_{|x|/4}(x))} \leq \|d_y\nabla V(y)\|_{L^\infty(B_{|x|/2}(x))} \leq 2C_5\left(1 + \|V\|_{L^\infty(\mathbb{R}^n)} + \|V\|^2_{L^\infty(\mathbb{R}^n)}\right).$$

Hence

$$\|\nabla V\|_{L^\infty(B_{\frac{|x|}{4}}(x))} \leq \frac{C_6}{|x|}\left(1 + \|V\|_{L^\infty(\mathbb{R}^n)} + \|V\|^2_{L^\infty(\mathbb{R}^n)}\right) \tag{3.9}$$

where $C_6 = 8C_5$. Let $r > 1$. Taking supremum over all $|x| \ge r$ in (3.9), we get

$$\|\nabla V\|_{L^\infty(\mathbb{R}^n\setminus B_r)} \le \frac{C_6}{r}\left(1 + \|V\|_{L^\infty(\mathbb{R}^n)} + \|V\|^2_{L^\infty(\mathbb{R}^n)}\right). \tag{3.10}$$

We next choose $\varepsilon_0 > 0$ such that for any $\sigma,\ \sigma' \in S^{n-1}$, the line segment $l_{\sigma,\sigma'}$ joining σ and σ' is outside $B_{\frac{1}{2}}$, i.e., $|\xi| \ge \frac{1}{2}$ for all $\xi \in l_{\sigma,\sigma'}$. Then by (3.10), $\forall x = r\sigma$, $x' = r\sigma'$, $|\sigma - \sigma'| < \epsilon_0$,

$$\begin{aligned}
|x\cdot\nabla v(x) - x'\cdot\nabla v(x')| =&|V(x) - V(x')|\\
=&\left|\int_0^1 \frac{\partial}{\partial t}V(tx + (1-t)x')\,dt\right|\\
\le&|x - x'|\cdot\sup_{0\le t\le 1}|\nabla V(tx + (1-t)x')|\\
\le&|\sigma - \sigma'|\cdot\sup_{0\le t\le 1}\frac{C}{t\sigma + (1-t)\sigma'}\\
\le&C|\sigma - \sigma'|
\end{aligned}$$

for some constant $C > 0$. Hence the family $\{rv_r(r\sigma)\}_{r>1}$ is equi-Hölder continuous on S^{n-1}. Let $\{r_i\}_{i=1}^\infty$ be a sequence of positive numbers such that $r_i \to \infty$ as $i \to \infty$. Then $\{r_i\}_{i=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly on S^{n-1} as $i \to \infty$. Then by the l'Hospital rule and Theorem 1.1,

$$\lim_{i\to\infty} r_i v_{r_i}(r_i\sigma) = \lim_{i\to\infty}\frac{v(r_i\sigma)}{\log r_i} = -2 \qquad \text{uniformly on } S^{n-1} \text{ as } i \to \infty.$$

Since the sequnece $\{r_i\}_{i=1}^\infty$ is arbitrary, $rv_r(r\sigma) \to -2$ uniformly on S^{n-1} as $r \to \infty$ and the proposition follows. □

We will now prove Theorem 1.3.

Proof of Theorem 1.3. Since any rotation in $\mathbb{R}^n$ can be decomposed into a finite number of rotations in 2-dimensional planes, it suffices to prove that

$$\Phi_{12} \equiv 0 \qquad \text{in } \mathbb{R}^n.$$

By direct computation Φ_{12} satisfies

$$\triangle q + \alpha e^v q + \beta\,(x\cdot\nabla v)\,e^v q + \beta\,(x\cdot\nabla q)\,e^v = 0.$$

Let $w(x) = \Phi_{12}(x)/g(x)$ where $g(x) = |x|^{2-n}$. Then w satisfies

$$\triangle w + \left(\frac{2\nabla g}{g} + \beta e^v x\right)\cdot\nabla w + \left(\alpha + \beta(x\cdot\nabla v) + \frac{\beta(x\cdot\nabla g)}{g}\right)e^v w = 0 \quad \text{in } \mathbb{R}^n.$$

If $\beta \le 0$, then $\alpha < n\beta \le 2\beta$. Then by Corollary 1.2 (1.1) has no solution and contradiction arises. Hence $\beta > 0$. We now choose $\varepsilon > 0$ such that $\alpha - n\beta + \epsilon\beta < 0$. By Proposition 3.2 there exists a constant $R_0 > 0$ such that

$$x\cdot\nabla v < -2 + \varepsilon \qquad \forall |x| \ge R_0.$$

Then

$$\alpha + \beta(x\cdot\nabla v) + \frac{\beta(x\cdot\nabla g)}{g} \le \alpha - n\beta + \varepsilon\beta < 0 \qquad \forall |x| \ge R_0$$

Suppose v is radially symmetric in B_{R_0}. Then $\Phi_{12} \equiv 0$ in $\overline{B_{R_0}}$. Hence

$$w \equiv 0 \qquad \text{in } \overline{B_{R_0}}.$$

We next observe that by (1.10),

$$|w| \le |x|^{n-2}|\Phi_{12}(x)| \to 0 \qquad \text{as } |x| \to \infty.$$

Then by the maximum principle for w in $R^n \backslash \overline{B_{R_0}}$,

$$w \equiv 0 \qquad \text{in } \mathbb{R}^n \backslash \overline{B_{R_0}}.$$

Hence

$$\Phi_{12} \equiv 0 \qquad \text{in } \mathbb{R}^n.$$

Thus v is radially symmetric in $\mathbb{R}^n$. □

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Kin Ming Hui: Institute of Mathematics, Academia Sinica,, Taipei, Taiwan, R.O.C.

E-mail address: kmhui@gate.sinica.edu.tw

Sunghoon Kim: Department of Mathematics and PMI (Pohang Mathematics Institute), Pohang University of Science and Technology (POSTECH),, Hyoja-Dong San 31, Nam-gu, Pohang 790-784, South Korea

E-mail address: math.s.kim@postech.ac.kr